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Extension of generalized plasticity to finite deformations and non-linear hardening[1]

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Abstract

The generalized plasticity model, as presented by Lubliner (1991), Lubliner et al. (1993) and Auricchio et al. (1992), is a generalization of classical rate-independent plasticity with a yield surface. This material model is able to describe a reloading transient during the reloading process of a specimen, which is shown in the asymptotic approach of the reloading curve to the initial load curve in the stress–strain diagram, as observed in Lubhahn and Felger (1961) for copper or in Greenstreet et al. (1971) for graphite. In the present paper an advantageous extension of the generalized plasticity model to finite plastic strain regimes is given. Classical plasticity and the model in Auricchio and Taylor (1995) for linear kinematic and isotropic hardening rules are included as special cases in the proposed concept. It will be shown that the suggested modification yields a mathematically simple structure of the constitutive relations and an efficient stress algorithm, adaptable to finite element programs similar to that in Auricchio and Taylor (1995). Special attention is focused on a correct treatment of the loading criteria.

Keywords: Finite deformation; Internal variable; Loading conditions; Modelling of asymptotic reloading; Plasticity

1. Introduction

In experiments with some materials, such as copper or graphite, it is observed in an unloading–reloading process that plasticity is renewed before the previous maximum stress level is attained, at which unloading began (see Fig. 1). This effect may be viewed as premature yielding and shall be denoted as *reloading transient* and may be accounted for by generalized plasticity.

The generalized plasticity model was presented in Lubliner (1991), Lubliner et al. (1993) and applied in Auricchio et al. (1992), Lubliner et al. (1993) and Auricchio and Taylor (1995). It is an extension of classical rate-independent plasticity with a yield surface. It may be argued that the reloading transient is a rate-dependent effect. However, the rate-independent generalized plasticity

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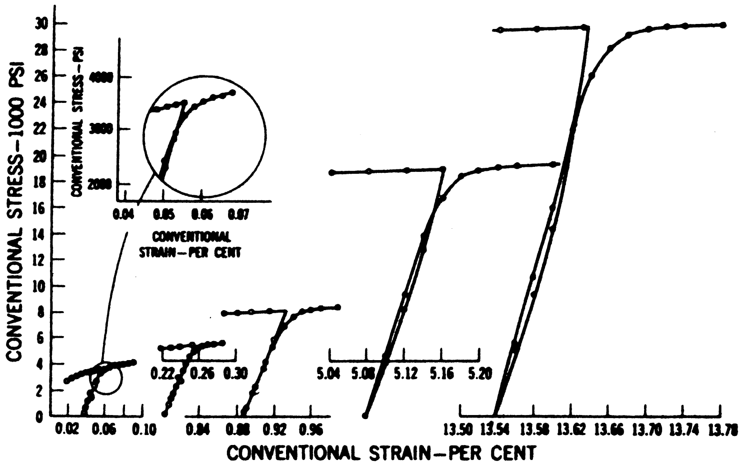

Fig. 1. Interrupted tensile test on certified OFHC copper, see Lubhahn et al. (1961, p. 486).

concept is in a position to model this phenomenon. The physical reasons for the reloading transient on reloading processes of previously unloaded rate-dependent materials is different to that for the reloading transient of rate-independent plastic materials. But the stress–strain diagrams of unloaded and reloaded rate-dependent materials are quantitatively similar to that of unloaded and reloaded rate-independent generalized plastic materials. The solution of boundary value problems with rate-dependent material models requires exact knowledge of the rate of loading during the entire process. The integration of rate-dependent constitutive equations must be performed during the complete actual time interval, which may be extremely long. Hence, the numerical analysis becomes laborious in some cases. The integration of the constitutive equations of rate-dependent generalized plasticity may be performed in *reduced time*, since time only orders the individual loading events in proper sequence, see Bland (1957). In addition, the time rate of loading has no influence on quasistatic mechanical processes for rate-independent material behaviour. If the rate of loading is known at all material points in advance, a rate-independent model with adequately chosen material parameters is sufficient for the solution of the associated boundary value problem.

In contrast to classical plasticity, where the yield function must vanish identically during plastic loading (see Simo, 1988), non-zero values of the yield criterion f are allowed in generalized plasticity. Following this new concept, the yield function may be regarded as an additional internal variable besides the plastic strains and the hardening variables, such as the backstresses and the plastic arclength. The missing constraint of a vanishing yield function necessitates an additional constitutive equation, termed as *limit equation* in Auricchio and Taylor (1995). Regarded as an evolution equation for the yield function, the limit equation is set up as a differential equation that governs the rate of change of the yield function during plastic flow. Of course, classical plasticity is comprised within this concept, if the yield function and its rate are set to zero throughout the plastic process.

The proposed extension of the generalized plasticity model to finite plastic strain regimes and non-linear hardening assumptions is based on a modification of the original limit equation for small strain plasticity of Auricchio and Taylor (1995):

$$F = h(f)[\tilde{N}\cdot\dot{\tilde{T}}] - \dot{\gamma} \leqslant 0. \tag{1.1}$$

The yield function is denoted as f, describing the yield surface in stress space, of which $\tilde{N}$ is the normal, rising in the current state of stress $\tilde{T}$. $h(f)$ is a function of f, $\dot{\tilde{T}}$ is the time derivative of the stress tensor $\tilde{T}$ and $\dot{\gamma}$ is the plastic multiplier. The scalar product $(\ )\cdot(\ )$ of two second-order tensors A and B is defined as $A\cdot B := \mathrm{tr}(AB^{\mathrm{T}})$, where tr( ) is the trace operator.

Problems with the integration of incomplete differentials are encountered with eqn (1.1), if non-linear kinematic hardening is included in generalized plasticity. The modified limit equation, given in the next section, avoids these difficulties. In Lubliner (1974, 1980a, 1980b, 1984, 1987, 1991), Auricchio et al. (1992), Lubliner et al. (1993) and Auricchio and Taylor (1995) the generalized plasticity model was presented for linear kinematic and isotropic hardening rules and small strains, although it was not restricted to these cases. A generalization of this model to non-linear hardening rules as well as finite strain regimes will be given without complicating the system of constitutive equations.

In Section 2.2 the concept of an unloaded elastic intermediate configuration is introduced, on which the constitutive equations are formulated. Limiting the application of finite strain elastoplasticity to finite plastic but small *elastic* strains, which is typical for many metal alloys, the system of constitutive non-linear equations, expressed with variables of the actual configuration, may be considerably simplified [compare eqns (2.5) and (3.2)].

The return mapping algorithms for the implementation into the finite element program FEAP will be presented for both regimes. The algorithms turn out to have a simple mathematical structure similar to that in Auricchio and Taylor (1995) and the system of non-linear constitutive equations may be reduced to the solution of only one scalar equation (see Section 4). Finally, numerical examples are presented in Section 5 to show the performance of the model.

2. Constitutive assumption for generalized plasticity

2.1. Yield function, loading criteria and limit equation

The yield function is given a key position in generalized plasticity. Some new aspects concerning the yield function in generalized plasticity are worked out in this section.

The mechanical state of an elastic body is entirely determined by the internal variables and the second Piola–Kirchhoff stresses $\tilde{T}$ or alternatively, the strains. The set of internal variables, considered in the present concept, encloses the plastic part of the strains, the plastic arclength s, the backstresses $\tilde{X}$ and the yield function f. Since the model shall be rate-independent, the variable time does not appear explicitly in the constitutive equations. The yield function considered is of the von Mises type:

$$f = \tilde{f}(\tilde{T}, \tilde{X}, s) = \Phi(\tilde{T} - \tilde{X}) - \sqrt{\tfrac{2}{3}}\,k(s) \lesseqgtr 0,$$

where $k(s)$ denotes the isotropic hardening variable. Let the backstresses, the plastic arclength and the yield function be given. All stress states belonging to the same value of the yield function f for $f \geqslant 0$ describe a convex five-dimensional hypersurface in six-dimensional stress space. This surface shall be denoted *yield surface* $\mathscr{F}$. The backstresses shift the yield surface in stress space, whereas

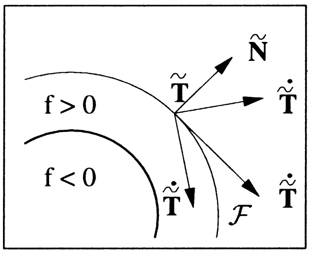

Fig. 2. Yield surface in stress space.

$f+\sqrt{\frac{2}{3}}k(s) = \Phi(\tilde{\boldsymbol{T}}-\tilde{\boldsymbol{X}}) \geqslant 0$ is a measure of its dilatation. The elastic domain is given by all states of stress where $f < 0$, i.e. all points with $f < 0$ may be accessed by an elastic (loading or unloading) process.

Let the yield surface $\mathscr{F}$ contain the actual state of stress and let the stress path be given. The normal to the yield surface in $\tilde{\boldsymbol{T}}$, pointing outward with respect to $\mathscr{F}$, shall be denoted by $\tilde{\boldsymbol{N}}$ and is calculated by

$$\tilde{\boldsymbol{N}} := \frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{T}}}, \quad f = \tilde{f}(\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{X}}, s).$$

The points inside $\mathscr{F}$ are always accessible by *elastic unloading*. This can be formulated as follows (see Fig. 2). The rate of stress $\dot{\tilde{\boldsymbol{T}}}$ may be viewed as a vector tangential to the stress-path. If this vector points inward with respect to the surface $\mathscr{F}$, then unloading occurs and $\tilde{\boldsymbol{N}} \cdot \dot{\tilde{\boldsymbol{T}}} < 0$ holds.

All points lying outside $\mathscr{F}$ are accessible if at all only by a loading process. A loading process happens accordingly, if $\tilde{\boldsymbol{N}} \cdot \dot{\tilde{\boldsymbol{T}}} \geqslant 0$. If $f < 0$, then loading produces elastic and if $f \geqslant 0$ plastic deformations. The loading criteria are summarised:

$$\begin{aligned} &f \geqslant 0 \quad \text{and} \quad \tilde{\boldsymbol{N}} \cdot \dot{\tilde{\boldsymbol{T}}} > 0 \rightarrow \text{plastic loading} \\ &f < 0 \quad \text{or} \quad \tilde{\boldsymbol{N}} \cdot \dot{\tilde{\boldsymbol{T}}} \leqslant 0 \rightarrow \text{elastic behaviour.} \end{aligned} \tag{2.1}$$

As long as $f \geqslant 0$ on loading or reloading paths, plastic flow occurs immediately. This is how the reloading transient is modelled by generalized plasticity. The yield function governs the loading condition through the definition of the elastic domain and the yield surface.

In the case of *plastic loading* the *limit equation*, introduced in Auricchio and Taylor (1995), governs the evolution of the yield function f, treated as an internal variable. Its general form shall be a rate-equation, homogeneous and of degree one in $\dot{s}$ for rate-independent plasticity. Since the asymptotic approach during repeated reloading processes may change with plastic deformations, the evolution equation of f shall depend in addition on the plastic arclength s.

$$\dot{f} = \hat{F}(s, \dot{s}, f). \tag{2.2}$$

In classical plasticity the yield function is not regarded as an internal variable, since it must vanish identically during plastic loading. Nevertheless, the yield function may be viewed to govern the rate of the internal variables, as the plastic arclength is calculated from the *consistency condition*

$\dot{f} = 0$. The special form of eqn (2.1) is chosen to encompass classical plasticity, which would not be comprised in generalized plasticity, if $f = 0$ would lead to elastic behaviour.

A convenient form of eqn (2.2), allowing closed form time integration for the solution, replaces eqn (1.1) of Auricchio and Taylor (1995) by:

$$F = h(f)\dot{f} - g(s)\dot{s} = 0, \tag{2.3}$$

with $h(f)$ and $g(s)$ as two arbitrary functions of the yield function f and plastic arclength s. Appropriate functions for $h(f)$ and $g(s)$ are discussed in Section 5, to model realistic material behaviour for the numerical examples presented.

2.2. Finite strain model

For the finite plastic strain model an unloaded intermediate configuration is introduced (see Fig. 3) and a multiplicative split of the deformation gradient into an elastic and a plastic part is assumed:

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p, \tag{2.4}$$

where the subscripts e and p indicate *elastic* and *plastic* contributions. It is observed from the transformation of GREEN's strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \mathbf{1})$ onto the intermediate configuration that the transformed strain tensor $\boldsymbol{\Gamma}$ is additively split into an elastic and a plastic part:

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_e + \boldsymbol{\Gamma}_p, \quad \boldsymbol{\Gamma}_e = \tfrac{1}{2}(\boldsymbol{F}_e^{\mathrm{T}}\boldsymbol{F}_e - \mathbf{1}), \quad \boldsymbol{\Gamma}_p = \tfrac{1}{2}(\mathbf{1} - \boldsymbol{F}_p^{-\mathrm{T}}\boldsymbol{F}_p^{-1}).$$

In the following all tensors related to the intermediate configuration are marked with capital letters, whereas those defined on the current configuration are marked with small letters. The decomposition in eqn (2.4) is determined only up to an orthogonal tensor $\boldsymbol{Q}$, i.e. $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \mathbf{1}$ and $\det \boldsymbol{Q} = 1$, since it holds:

$$\boldsymbol{F} = \boldsymbol{F}_e\boldsymbol{F}_p = (\boldsymbol{F}_e\boldsymbol{Q}^{\mathrm{T}})(\boldsymbol{Q}\boldsymbol{F}_p) =: \boldsymbol{F}_e^{+}\boldsymbol{F}_p^{+}.$$

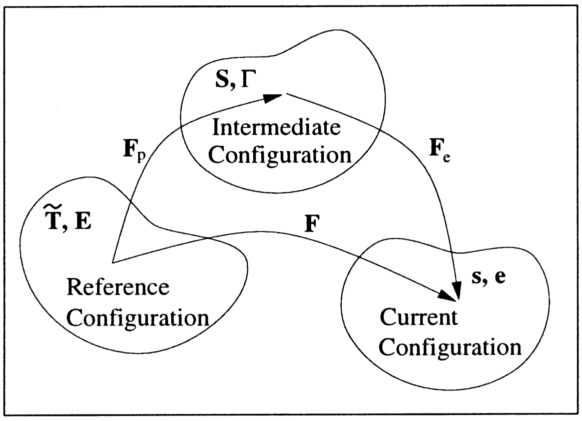

Fig. 3. Configurations and tensors.

For convenience the *postulate of invariance* is adopted, i.e. the constitutive equations, expressed with variables on the intermediate configuration, contain no information about the rotation of the intermediate configuration (see Casey and Naghdi, 1980). Firstly, it is observed, that the plastic right CAUCHY–GREEN tensor $\boldsymbol{C}_p$ is independent of $\boldsymbol{Q}$:

$$\boldsymbol{C}_p = \boldsymbol{F}_p^{\mathrm{T}}\boldsymbol{F}_p = \boldsymbol{F}_p^{+\mathrm{T}}\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}_p^{+} = \boldsymbol{F}_p^{+\mathrm{T}}\boldsymbol{F}_p^{+} =: \boldsymbol{C}_p^{+},$$

that is any function of $\boldsymbol{C}_p$ or equivalently $\boldsymbol{\Gamma}_p$ fulfills the postulate of invariance. Secondly, any function of $\boldsymbol{C}_e = \boldsymbol{F}_e^{\mathrm{T}}\boldsymbol{F}_e$ must be isotropic in $\boldsymbol{C}_e$. This restricts the elasticity relation to isotropy as is shown next. Let $\boldsymbol{S}(\boldsymbol{C}_e)$ be the stress tensor, resulting from the transformation of the KIRCHHOFF-stress tensor $\boldsymbol{s}$ onto the intermediate configuration. Then, by the postulate of invariance, it must hold:

$$\boldsymbol{s} = \boldsymbol{F}_e\boldsymbol{S}(\boldsymbol{C}_e)\boldsymbol{F}_e^{\mathrm{T}} = \boldsymbol{F}_e^{+}\boldsymbol{Q}\boldsymbol{S}(\boldsymbol{C}_e)\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}_e^{+\mathrm{T}} \stackrel{!}{=} \boldsymbol{F}_e^{+}\boldsymbol{S}(\boldsymbol{C}_e^{+})\boldsymbol{F}_e^{+\mathrm{T}}$$
$$\rightarrow \boldsymbol{S}(\boldsymbol{C}_e^{+}) = \boldsymbol{Q}\boldsymbol{S}(\boldsymbol{C}_e)\boldsymbol{Q}^{\mathrm{T}},$$

where $\boldsymbol{C}_e^{+} = \boldsymbol{F}_e^{+\mathrm{T}}\boldsymbol{F}_e^{+}$. The isotropic, non-linear elasticity relation of Simo and Pister (1984) satisfies this requirement and is used to model the elastic response:

$$\boldsymbol{S} = K\ln\left[\det(\boldsymbol{C}_e)^{1/2}\right]\boldsymbol{C}_e^{-1} + G\det(\boldsymbol{C}_e)^{-1/3}\left[\mathbf{1} - \tfrac{1}{3}\operatorname{tr}(\boldsymbol{C}_e)\boldsymbol{C}_e^{-1}\right]. \tag{2.5}$$

Herein, K and G are material parameters, comparable to the bulk and shear modulus. In the context of finite strains eqn (2.5) shows a physically meaningful stress–strain response.

The yield criterion in terms of the stresses $\boldsymbol{S}$ is given by:

$$f = \hat{f}(\boldsymbol{S}, \boldsymbol{X}, s) = \|(\boldsymbol{S}-\boldsymbol{X})^{\mathrm{D}}\| - \sqrt{\tfrac{2}{3}}k(s) \leqslant 0, \tag{2.6}$$

where the bracket $\| \ \|$ denotes the EUCLIDean norm, e.g. $\|\boldsymbol{S}\| = \sqrt{\boldsymbol{S}\cdot\boldsymbol{S}}$ and $(\cdot)^{\mathrm{D}}$ the deviator. The time derivative of the yield function in terms of the variables on the intermediate configuration is:

$$\dot{f} = \boldsymbol{N}\cdot(\dot{\boldsymbol{S}} - \dot{\boldsymbol{X}}) - \sqrt{\tfrac{2}{3}}k'(s)\dot{s},$$

with $k'(s) = (\partial k/\partial s)$ and $\boldsymbol{N} = (\partial\hat{f}/\partial\boldsymbol{S}) = [(\boldsymbol{S}-\boldsymbol{X})^{\mathrm{D}}/\|(\boldsymbol{S}-\boldsymbol{X})^{\mathrm{D}}\|]$. Before the evolution equations of the tensor valued internal variables are specified, objective time derivatives of the second-order tensors, defined on the time dependent intermediate configuration, must be given. By formulation of the constitutive equations with those objective time derivatives, no information about the rotation of the intermediate configuration will be introduced into the physically relevant variables defined on the reference or actual configuration (see for example Appendix A).

The concept of dual variables in Haupt and Tsakmakis (1989) is applied, where apart from the stress power also the incremental stress power is invariant under a special group of transformations, namely the same rules of transformation for stress and strain rates between the initial and the intermediate configuration shall hold as they do for stress and strain tensors. This leads to the OLDROYD derivatives for the chosen pair of dual variables, given for the covariant plastic strain tensor $\boldsymbol{\Gamma}_p$ as:

$$\overset{\Delta_p}{\boldsymbol{\Gamma}}_p = \boldsymbol{F}_p^{-\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{F}_p^{\mathrm{T}}\boldsymbol{\Gamma}_p\boldsymbol{F}_p)\boldsymbol{F}_p^{-1} = \dot{\boldsymbol{\Gamma}}_p + \boldsymbol{L}_p^{\mathrm{T}}\boldsymbol{\Gamma}_p + \boldsymbol{\Gamma}_p\boldsymbol{L}_p$$

and for the contravariant backstress tensor $\boldsymbol{X}$ and the stress tensor $\boldsymbol{S}$ as:

$$\overset{\nabla_p}{\boldsymbol{X}} = \boldsymbol{F}_p \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{F}_p^{-1}\boldsymbol{X}\boldsymbol{F}_p^{-\mathrm{T}})\boldsymbol{F}_p^{\mathrm{T}} = \dot{\boldsymbol{X}} - \boldsymbol{L}_p\boldsymbol{X} - \boldsymbol{X}\boldsymbol{L}_p^{\mathrm{T}},$$

$$\overset{\nabla_p}{\boldsymbol{S}} = \boldsymbol{F}_p \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{F}_p^{-1}\boldsymbol{S}\boldsymbol{F}_p^{-\mathrm{T}})\boldsymbol{F}_p^{\mathrm{T}} = \dot{\boldsymbol{S}} - \boldsymbol{L}_p\boldsymbol{S} - \boldsymbol{S}\boldsymbol{L}_p^{\mathrm{T}}, \tag{2.7}$$

defined both on the intermediate configuration and with $\boldsymbol{L}_p = \dot{\boldsymbol{F}}_p\boldsymbol{F}_p^{-1}$ as the velocity gradient, formed with the plastic part of the deformation gradient.

By making use of the identity (see Appendix A)

$$\boldsymbol{N}\cdot\overset{\nabla_p}{\boldsymbol{S}} = \tilde{\boldsymbol{N}}\cdot\dot{\tilde{\boldsymbol{T}}},$$

the loading criteria in eqn (2.1) are extended to finite deformation plasticity:

$$f \geqslant 0 \quad \text{and} \quad \boldsymbol{N}\cdot\overset{\nabla_p}{\boldsymbol{S}} > 0 \rightarrow \text{plastic loading}$$

$$f < 0 \quad \text{or} \quad \boldsymbol{N}\cdot\overset{\nabla_p}{\boldsymbol{S}} \leqslant 0 \rightarrow \text{elastic behaviour}. \tag{2.8}$$

The plastic strains shall evolve according to the normality rule:

$$\overset{\Delta_p}{\boldsymbol{\Gamma}}_p = \begin{cases} \dot{\gamma}\dfrac{\partial\hat{f}}{\partial\boldsymbol{S}} = \dot{\gamma}\boldsymbol{N} & \text{for plastic loading} \\ \boldsymbol{0} & \text{otherwise} \end{cases}, \quad \boldsymbol{\Gamma}_p(t=0) = \boldsymbol{0}.$$

The normal $\boldsymbol{N}$ is a deviatoric second-order tensor, which implies isochoric inelastic strains:

$$0 = \det(\boldsymbol{F}_p)\,\mathrm{tr}(\overset{\Delta_p}{\boldsymbol{\Gamma}}_p) = \det(\boldsymbol{F}_p)\,\mathrm{tr}(\boldsymbol{L}_p) = \det(\boldsymbol{F}_p)\boldsymbol{F}_p^{-\mathrm{T}}\cdot\dot{\boldsymbol{F}}_p = \frac{\mathrm{d}}{\mathrm{d}t}\det(\boldsymbol{F}_p). \tag{2.9}$$

2.3. On models without an elastic domain

According to eqns (2.6) and (2.8) the elastic domain vanishes for some materials, e.g. graphite, if $k = 0$, since $f = \|(\boldsymbol{S}-\boldsymbol{X})^{\mathrm{D}}\|$ is never negative. In classical plasticity with $f = k = 0$, the normal can no more be calculated, as the quantity $(\boldsymbol{S}-\boldsymbol{X})^{\mathrm{D}}$ becomes the zero tensor. Hence, any deformation is plastic. The plastic strain would be equal to the total strain deviator and the evolution equation for kinematic hardening with the identity

$$\boldsymbol{S}^{\mathrm{D}} = \boldsymbol{X}^{\mathrm{D}}$$

yields a system of six differential equations for the stresses.

In generalized plasticity the problem of a vanishing elastic domain may be solved elegantly. Only in the initial state where $f = 0$ the normal cannot be calculated. As soon as the yield function becomes positive the normal is defined, since the yield surface is no more degenerated to a single point. Therefore, it is suggested, to modify the loading conditions in eqn (2.8) in case of $k \equiv 0$ in such a way, that in the initial state with $f = 0$ elastic deformations occur:

$$f > 0 \quad \text{and} \quad \boldsymbol{N} \cdot \overset{\nabla_p}{\boldsymbol{S}} > 0 \rightarrow \text{plastic loading}$$

$$f = 0 \quad \text{or} \quad \boldsymbol{N} \cdot \overset{\nabla_p}{\boldsymbol{S}} \leqslant 0 \rightarrow \text{elastic behaviour.}$$

Note the difference to the concept in Dafalias (1977). There a *bounding surface* was introduced to define the normal. In general, the bounding surface does not include the stress state, which must be projected onto the bounding surface to find the point, where the normal can be computed. In generalized plasticity the stress state is by definition on the yield surface $f \geqslant 0$, where the normal may be calculated (see also Lubliner, 1991).

2.4. Generalized plasticity with hardening

The plastic arclength s and the isotropic hardening variable $k(s)$ are given as

$$\dot{s} = \sqrt{\tfrac{2}{3}}\,\|\overset{\Delta_p}{\boldsymbol{\Gamma}}_p\|, \quad s(t=0) = 0$$

$$k(s) = k_0 + k_l s + k_e\,\mathrm{e}^{-\alpha s},$$

where k_0, k_l, k_e and α are material parameters. The constitutive equation for the evolution of the backstresses $\boldsymbol{X}$ is of the Armstrong and Frederick type (see Armstrong and Frederick, 1966):

$$\overset{\nabla_p}{\boldsymbol{X}} = c\overset{\Delta_p}{\boldsymbol{\Gamma}}_p - b\dot{s}\boldsymbol{X}, \quad \boldsymbol{X}(t=0) = \boldsymbol{0},$$

where b and c are material parameters.

3. Transformation to the current configuration

The transformation of tensorial variables from the elastic intermediate configuration to the current configuration is performed by means of the elastic part $\boldsymbol{F}_e$ or the deformation gradient with the transformation rules for covariant tensors

$$\boldsymbol{y} = \boldsymbol{F}_e^{-\mathrm{T}}\boldsymbol{Y}\boldsymbol{F}_e^{-1}$$

and contravariant tensors

$$\boldsymbol{z} = \boldsymbol{F}_e\boldsymbol{Z}\boldsymbol{F}_e^{\mathrm{T}}. \tag{3.1}$$

The above indicated convention for small and capital letter symbols for tensors still holds. The system of constitutive equations may be considerably simplified by the assumption of finite plastic strains but small elastic strains. As shown by the polar decomposition theorem

$$\boldsymbol{F}_e = \boldsymbol{R}_e \boldsymbol{U}_e = \boldsymbol{R}_e\sqrt{\boldsymbol{C}_e} = \boldsymbol{R}_e\sqrt{\mathbf{1}+2\boldsymbol{\Gamma}_e},$$

small elastic strains, i.e. $\|\boldsymbol{\Gamma}_e\| \ll 1$, imply approximately:

$$\boldsymbol{F}_e \approx \boldsymbol{R}_e.$$

In the case of small elastic deformations, the elastic deformation gradient is approximated by its finite rotation part. The constitutive equations, expressed in terms of variables of the current configuration, simplify with this approximation to

$$\boldsymbol{s} = K\ln(\det \boldsymbol{F})\mathbf{1} + G\boldsymbol{b}_e^{\mathrm{D}}, \tag{3.2}$$

where $\boldsymbol{b}_e$ is the left Cauchy–Green tensor equal to $\boldsymbol{F}_e\boldsymbol{F}_e^{\mathrm{T}}$, formed by the elastic deformation gradient. Hence,

$$\sqrt{\det \boldsymbol{b}_e} = \det \boldsymbol{F}_e = \det \boldsymbol{F},$$

where the last equality follows from plastic incompressibility due to eqns (2.6) and (2.9). The yield function f reduces approximately to $\bar{f}$ [see Appendix B, eqn (B.2)]

$$f \approx \bar{f}(\boldsymbol{s}, \boldsymbol{x}, s) = \|(\boldsymbol{s}-\boldsymbol{x})^{\mathrm{D}}\| - \sqrt{\tfrac{2}{3}}k(s) = \|\boldsymbol{\sigma}\| - \sqrt{\tfrac{2}{3}}k(s) \tag{3.3}$$

with

$$\boldsymbol{\sigma} = (\boldsymbol{s}-\boldsymbol{x})^{\mathrm{D}} \quad \text{and} \quad \bar{\boldsymbol{n}} = \frac{\partial \bar{f}}{\partial \boldsymbol{s}} = \frac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|}.$$

The OLDROYD derivative, applied in the following to tensor variables with base vectors on the current configuration, are defined for co- and contravariant second-order tensors ( ) as:

$$\overset{\triangledown}{()} = \dot{()} + \boldsymbol{L}^{\mathrm{T}}() + ()\boldsymbol{L},$$
$$\overset{\vartriangle}{()} = \dot{()} - \boldsymbol{L}() - ()\boldsymbol{L}^{\mathrm{T}}, \tag{3.4}$$

where $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$ is the velocity gradient. It may be shown (see Appendix B) that the loading criterion in terms of tensors, defined on the current configuration, becomes:

$$\bar{f} \geqslant 0 \quad \text{and} \quad \bar{\boldsymbol{n}}\cdot\overset{\triangledown}{\boldsymbol{s}} > 0 \rightarrow \text{plastic loading}$$
$$\bar{f} < 0 \quad \text{or} \quad \bar{\boldsymbol{n}}\cdot\overset{\triangledown}{\boldsymbol{s}} \leqslant 0 \rightarrow \text{elastic behaviour}. \tag{3.5}$$

The limit equation of (2.3) in terms of tensors on the current configuration retains its simple mathematical structure due to the similarity of eqns (2.6) and (3.3), whose total differentials yields:

$$\dot{f} - \sqrt{\frac{2}{3}}k'(s)\dot{s} = \frac{\partial \hat{f}}{\partial \boldsymbol{S}}\cdot(\dot{\boldsymbol{S}} - \dot{\boldsymbol{X}}) \approx \frac{\partial \bar{f}}{\partial \boldsymbol{s}}\cdot(\dot{\boldsymbol{s}} - \dot{\boldsymbol{x}}).$$

The evolution equations of the plastic part of the ALMANSI strain tensor $\boldsymbol{e}_p$ and the backstress tensor $\boldsymbol{x}$ on the current configuration (see Lührs et al., 1997 for more details) take the form:

$$\overset{\Delta}{\boldsymbol{e}}_p = \begin{cases} -\frac{1}{2}\overset{\nabla}{\boldsymbol{b}}_e = \dot{\gamma}\bar{\boldsymbol{n}} & \text{for plastic loading} \\ \boldsymbol{0} & \text{otherwise} \end{cases}, \quad \boldsymbol{e}_p(t=0) = \boldsymbol{0} \tag{3.6}$$

$$\dot{s} = \sqrt{\tfrac{2}{3}}\,\|\overset{\Delta}{\boldsymbol{e}}_p\|, \quad s(t=0) = 0 \tag{3.7}$$

$$\overset{\nabla}{\boldsymbol{x}} = c\overset{\Delta}{\boldsymbol{e}}_p - b\dot{s}\boldsymbol{x} = \dot{\gamma}(c\bar{\boldsymbol{n}} - \sqrt{\tfrac{2}{3}}\,b\boldsymbol{x}), \quad \boldsymbol{x}(t=0) = \boldsymbol{0}. \tag{3.8}$$

4. Numerical solution of the constitutive equations

From the known process variables ${}^n\boldsymbol{u}$, ${}^n\boldsymbol{b}_e$, ${}^n\boldsymbol{x}$, ${}^ns$ and ${}^n\bar{f}$ in the equilibrium configuration at time ${}^nt$, their new values at ${}^{n+1}t$ are to be calculated for a given displacement increment $\Delta\boldsymbol{u} = {}^i\boldsymbol{u} - {}^n\boldsymbol{u}$ in the context of standard finite element procedures. If the stresses satisfy the momentum balance equations within a small tolerance, the history variables $\boldsymbol{x}$, $\boldsymbol{b}_e$, s and $\bar{f}$ are updated by replacing ${}^n\boldsymbol{x}$, ${}^n\boldsymbol{b}_e$, ${}^n\boldsymbol{u}$, ${}^ns$ and ${}^n\bar{f}$ by ${}^m\boldsymbol{x}$, ${}^m\boldsymbol{b}_e$, ${}^m\boldsymbol{u}$, ${}^ms$ and ${}^m\bar{f}$ of the last (m-th) iteration for time ${}^{n+1}t$.

The incremental deformation gradient $\boldsymbol{F}_u$ is computed from the displacement fields ${}^i\boldsymbol{u}$ in the i-th iteration and ${}^n\boldsymbol{u}$ of the last equilibrium position:

$$\boldsymbol{F}_u = {}^i\boldsymbol{F}\,{}^n\boldsymbol{F}^{-1}. \tag{4.1}$$

The stresses ${}^i\boldsymbol{s}$ at time ${}^it$ are calculated with the *radial return* algorithm. In the *elastic-predictor* step the trial state with italic superscript T on variables is defined by assuming elastic behaviour with the plastic configuration 'fixed', i.e. $\dot{\boldsymbol{F}}_p = \boldsymbol{0}$. The internal variables of tensorial character at the last equilibrium state are transformed onto the current configuration according to

$$ {}^T\boldsymbol{b}_e = \boldsymbol{F}_u\,{}^n\boldsymbol{b}_e\boldsymbol{F}_u^{\mathrm{T}} \quad \text{and} \quad {}^T\boldsymbol{x} = \boldsymbol{F}_u\,{}^n\boldsymbol{x}\boldsymbol{F}_u^{\mathrm{T}}. \tag{4.2}$$

The trial stresses are received by inserting ${}^T\boldsymbol{b}_e$ and ${}^i\boldsymbol{F}$ into the elasticity relation in eqn (3.2):

$$ {}^T\boldsymbol{s} = K\ln(\det{}^i\boldsymbol{F})\boldsymbol{1} + G\,{}^T\boldsymbol{b}_e^{\mathrm{D}}.$$

In the trial state the plastic arclength remains constant with respect to the last equilibrium state, i.e. ${}^Ts = {}^ns$. The yield function ${}^T\bar{f}$, regarded as a history variable, becomes:

$$ {}^T\bar{f} = \|({}^T\boldsymbol{s} - {}^T\boldsymbol{x})^{\mathrm{D}}\| - \sqrt{\tfrac{2}{3}}\,k({}^Ts).$$

The term $\bar{\boldsymbol{n}}\cdot\overset{\nabla}{\boldsymbol{s}} = (\partial\bar{f}/\partial\boldsymbol{s})\cdot\overset{\nabla}{\boldsymbol{s}}$ in the loading conditions of eqn (3.5) is approximated in the elastic predictor step with the time increment $\Delta t = {}^{n+1}t - {}^nt$ by the difference quotient (see Appendix C):

$$\bar{\boldsymbol{n}}\cdot\overset{\nabla}{\boldsymbol{s}} \approx \frac{{}^T\bar{f} - {}^n\bar{f}}{\Delta t} =: \frac{\Delta\bar{f}}{\Delta t}. \tag{4.3}$$

If either

$$ {}^T\bar{f} \leqslant 0 \quad \text{or} \quad \Delta\bar{f} \leqslant 0,$$

the assumption of an elastic process holds and the solution for this iterate at time ${}^{n+1}t = {}^it$ is found to be the result of the trial state.

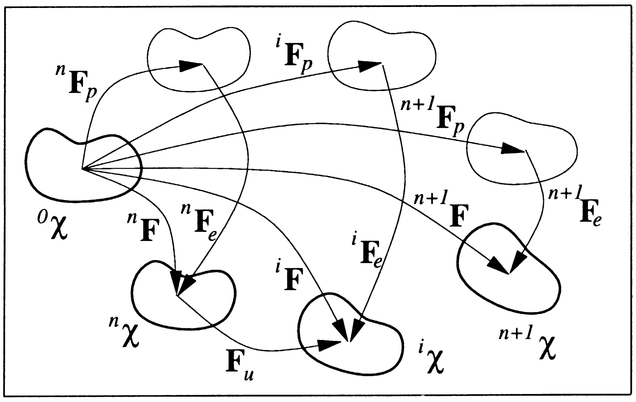

Fig. 4. Configurations during the iteration.

If *not*, the *plastic corrector* has to be computed in the second step, where the deformation is kept constant, i.e. $\dot{\boldsymbol{F}} = \boldsymbol{0}$ and thus, the velocity gradient $\boldsymbol{L}$ vanishes. The OLDROYD derivatives turn into the material time derivatives:

$$\overset{\nabla}{\boldsymbol{b}}_e = \dot{\boldsymbol{b}}_e, \quad \overset{\nabla}{\boldsymbol{x}} = \dot{\boldsymbol{x}} \quad \text{and} \quad \overset{\nabla}{\boldsymbol{s}} = \dot{\boldsymbol{s}}. \tag{4.4}$$

The evolution equations are numerically integrated in time by means of a backward EULER-difference scheme. The mathematical structure of the radial return mapping algorithm of the infinitesimal theory is conserved by discretizing the inverse of the plastic deformation tensor, see Simo (1988):

$$\begin{aligned} \boldsymbol{C}_p &= \boldsymbol{F}_p^{\mathrm{T}}\boldsymbol{F}_p = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{b}_e^{-1}\boldsymbol{F} \\ {}^{i}\boldsymbol{C}_p^{-1} &= {}^{n}\boldsymbol{C}_p^{-1} + \Delta t\dot{\boldsymbol{C}}_p^{-1}|_{{}^{i}t}. \end{aligned} \tag{4.5}$$

Pre- and post-multiplying eqn (4.5) by ${}^{i}\boldsymbol{F}$ resp. ${}^{i}\boldsymbol{F}^{\mathrm{T}}$ and making use of the time derivative of eqn (4.5) $\dot{\boldsymbol{b}}_e = \boldsymbol{F}\dot{\boldsymbol{C}}_p^{-1}\boldsymbol{F}^{\mathrm{T}}$ for $\dot{\boldsymbol{F}} = \boldsymbol{0}$ leads to:

$${}^{i}\boldsymbol{b}_e = {}^{i}\boldsymbol{F}\,{}^{n}\boldsymbol{F}^{-1}\,{}^{n}\boldsymbol{b}_e\,{}^{n}\boldsymbol{F}^{-\mathrm{T}}\,{}^{i}\boldsymbol{F}^{-\mathrm{T}} + \Delta t\dot{\boldsymbol{b}}_e|_{{}^{i}t}.$$

With the definitions in eqns (4.1) and (4.2) this simplifies to:

$${}^{i}\boldsymbol{b}_e = {}^{T}\boldsymbol{b}_e + \Delta t\dot{\boldsymbol{b}}_e|_{{}^{i}t} \tag{4.6}$$

as postulated in Lührs et al. (1997), eqn (5.12). Equation (3.6) is inserted into eqn (4.6):

$${}^{i}\boldsymbol{b}_e = {}^{T}\boldsymbol{b}_e - 2\lambda\,{}^{i}\bar{\boldsymbol{n}} + \varsigma\boldsymbol{1}, \tag{4.7}$$

where the plastic rate parameter is defined as $\lambda = \Delta t\dot{\gamma}$ and a spherical part $\varsigma\boldsymbol{1}$ is added in to ensure the plastic incompressibility condition, since eqn (2.9) is not satisfied exactly for finite time increments. The scalar ς, subjected to the constraint:

$$\det{}^2({}^{i}\boldsymbol{F}) = \det({}^{i}\boldsymbol{b}_e) = \det({}^{T}\boldsymbol{b}_e - 2\lambda\,{}^{i}\bar{\boldsymbol{n}} + \varsigma\boldsymbol{1})$$

need only be calculated after equilibrium is obtained (see Lührs et al., 1997). Substitution of eqn (4.7) into eqn (3.2) results in

$$ {}^{i}\boldsymbol{s} = {}^{T}\boldsymbol{s} - 2G\lambda\, {}^{i}\bar{\boldsymbol{n}} \tag{4.8} $$

and eqns (3.6) and (4.4) into the discretized equations (3.7) and (3.8) gives:

$$ {}^{i}\boldsymbol{x} = T({}^{T}\boldsymbol{x} + \lambda c\, {}^{i}\bar{\boldsymbol{n}}) \tag{4.9} $$

$$ {}^{i}s = {}^{n}s + \surd $$